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LETTER TO THE EDITOR

Random walks on hierarchical lattices at the percolation threshold

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Abstract. Random walks on hierarchical lattices are considered at the percolation threshold. Real-space renormalisation methods are employed to obtain the exact fractal dimension of the incipient infinite cluster (IIC), d_f, and the fractal dimension of the effective one-dimensional resistance length of the IIC, d_r, for some hierarchical lattices. The recent conjecture of Aharony and Stauffer does not hold for diamond and Wheatstone bridge hierarchical lattices.

In studies of random walks on percolation clusters at the percolation threshold, a remarkable universality was conjectured by Alexander and Orbach (1982) which states that $d_w = \frac{3}{2}d_f$, for $d \geq 2$, where d_w is the random walk dimension and d_f is the fractal dimension of the substrate. Early numerical works (Ben-Abraham and Havlin 1983, Pandey and Stauffer 1983) as well as ε expansion (Dasgupta *et al* 1977) supported the Alexander and Orbach (AO) conjecture. However, recently, arguments against the AO conjecture began to appear in the literature (Coniglio and Stanley 1984, Family and Coniglio 1984, Harris *et al* 1984). In particular the most accurate numerical works of four different groups reached the conclusion that the AO conjecture definitely fails in $d=2$ (Zabolitzky 1984, Herrmann *et al* 1984, Hong *et al* 1984a, Lobb and Frank 1984). The failure of the AO conjecture implies that the frontier sites of the walker (or growth sites in the terminology of Leyvraz and Stanley 1983) is not proportional to $(S_N)^{-1/2}$, where S_N is the number of sites visited by the walker at time step N (Rammal and Toulouse 1983, Leyvraz and Stanley 1983, Alexander 1983, Ben-Abraham and Havlin 1983).

In order to understand the failure of the AO conjecture in percolation, Aharony and Stauffer (1984) have recently proposed the startling conjecture that for any random fractal with $d_f \leq 2$‡, one has $d_w = d_f + 1$. The physical idea behind this is that the frontier sites of the walker are on the sphere cut of radius R (see also Alexander 1983). Using the relation $d_w = d_f + d_r$ (Alexander and Orbach 1982, Stanley and Coniglio 1984)—where d_r is the fractal dimension of the effective one-dimensional resistance length of IIC—one has from Aharony and Stauffer (AS) that $d_r = 1$ if $d_f \leq 2$. This is intriguing and calls for a direct test.

In this letter we construct random fractals whose fractal dimension is less than two and obtain the exact value d_r. The result serves to test the validity of the AS conjecture in random fractals.

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‡ The proposal of Aharony and Stauffer (AS) that the lower critical dimension of the AO conjecture is $d_f = 2$ was independently proposed by Coniglio and Stanley (1984).

Consider a diamond hierarchical lattice (figure (1*a*)) that is constructed by an iterative generation of the base set. Each bond is occupied with probability p and absent with $1-p$. For the nth order generation, the recursion relation for P_n, the occupation probability, satisfies (Kaufman and Andelman 1984)

$$P_{n+1} = 2P_n^2 - P_n^4, \tag{1}$$

which has two trivial fixed points $p = 0, 1$ and one non-trivial point $P^* = (\sqrt{5}-1)/2 = 0.618$, which governs the percolating phase. In the vicinity of the percolation threshold, we define the fractal dimension of the IIC through

$$M \sim \xi^{d_f}, \tag{2}$$

where M is the mass of IIC and ξ is the correlation length. Let the mass and correlation length of the nth generation be M_n and ξ_n respectively. Then under rescaling transformation, M_n and M_{n-1} satisfy

$$(M_{n-1}/M_n) = (\xi_{n-1}/\xi_n)^{d_f}. \tag{3}$$

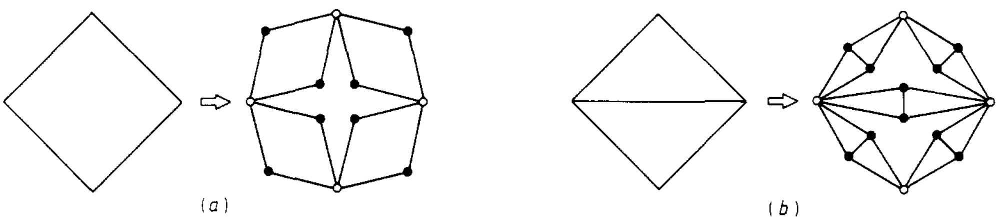

Figure 1. Construction of (*a*) diamond and (*b*) Wheatstone bridge hierarchical lattices by decoration (see Berker and Ostlund 1979, Griffiths and Kaufman 1981, and Kaufman and Griffiths 1984). Black sites are decimated upon renormalisation.

Therefore the fractal dimension d_f becomes

$$d_f = \log(M_{n-1}/M_n)/\log(\xi_{n-1}/\xi_n). \tag{4}$$

Clearly $\xi_{n-1}/\xi_n = 2$†. Since mass is the sum of total bonds, M_n satisfies the obvious linear scaling relation

$$M_n(\lambda) = \lambda M_n(1), \tag{5}$$

where 1 is the mass of each bond and λ is an arbitrary scale parameter.

After decimation (see figure 1), we have (Hong and Stanley 1983)

$$d_f = \log(\lambda_t/p^*)/\log 2, \tag{6}$$

where λ_t is the mean number of bonds connecting the top and bottom of the base. With bond occupation probability p, λ_t satisfies

$$\lambda_t = 4p^4 + 12p^3q + 4p^2q^2|_{p=p^*} = 1.888. \tag{7}$$

Substituting (7) into (4), we obtain $d_f = 1.611$, which is less than 2.

† The definition of distance and correlation length on the hierarchical lattices is not clear. However if we define the minimum path as the distance between two points, then the usual definition of correlation length in critical phenomena holds on the hierarchical lattices as well (see Berker and Ostlund 1979, Griffith and Kaufman 1981, 1982, Kaufman and Griffith 1984).

Now we turn our attention to the calculation of the fractal dimension of the effective one-dimensional resistance length L_r of IIC defined through

$$L_r \sim \xi^{d_r}. \tag{8}$$

$L_r^{(n)}$, which is the L_r of the nth generation, also satisfies the linear scaling relation

$$L_r^{(n)}(\lambda\rho) = \lambda L_r^{(n)}(\rho), \tag{9}$$

where λ is an arbitrary scale variable and ρ is a resistance of one bond. It follows upon decimation that

$$d_r = \log(\lambda_r/p^*)/\log 2, \tag{10}$$

where λ_r is the mean resistance of the base at $p = p^*$. Using $\lambda_r = -3p^4 + 4p^2|_{p=p^*} = 1.090$, we arrive at $d_r = 0.8189$, which is not in accord with the AS conjecture. We have also calculated d_r for a Wheatstone bridge hierarchical lattice (figure 1(b)) and we find that $d_f = 1.71$ and $d_r = 0.9386$, which again is less than 1. Therefore, at least in the case of diamond and Wheatstone bridge hierarchical lattices, the AS conjecture fails. It is conceivable that the AS conjecture does not hold in other hierarchical lattices either.

We conclude with several observations. First, using the relation $d_w = d_f + d_r$, one can calculate the fracton dimension $\bar{\bar{d}} = 2d_f/d_w$. For a diamond hierarchy we get $\bar{\bar{d}} = 1.326$, and for a Wheatstone bridge hierarchy we have $\bar{\bar{d}} = 1.296$. Note that the fracton dimension for both cases is close to $\frac{4}{3}$. Second, since the magnetic scaling power Y_h is the same as d_f (Stanley 1977, Hong *et al* 1984b, c), the Potts model in the limit $q \to 1$ can be used to obtain the fractal dimension of IIC. However, due to the heterogeneity of the coordinates, the treatment of external fields in the hierarchical lattices is awkward (Melrose 1983). The fractal dimension serves instead to define Y_h unambiguously in the limit $q \to 1$ on the hierarchical lattices.

In summary, we have used the simple real-space renormalisation method to obtain the exact fractal dimension of IIC and the effective one-dimensional resistance length of IIC as well as the fracton dimension for the families of hierarchical lattices. Results are not consistent with the recent conjecture of Aharony and Stauffer.

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